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DETERMINATION OF THE RING-SHAPED MASS DISTRIBUTION GRAVITATIONAL FIELD STRENGTH

The case of a ring-shaped mass distribution is considered. A method for estimating the gravitational field strength magnitude at points localized on the ring is investigated, taking into account the finiteness of the volumetric mass density. There is obtained the expression that allows us to estimate the gravitational field strength magnitude at the points localized on the ring.

Keywords: gravitational field strength, ring radius, ring mass.

Introduction. There are planets in the solar system surrounded by rings. They consist of cosmic dust and ice. Saturn was the first planet to have rings discovered. The total mass of debris in the ring system is estimated to be $(1{,}54 \pm 0.49) \times 10^{19}$ kg.

From photographs taken by the automatic interplanetary station «Voyager 1», it was established that the rings of Saturn consist of hundreds of narrow rings. Subsequently, the images obtained by the probes show that in fact the rings are formed from thousands of rings alternating with slits.

It is interesting to note that interest in the rings of Saturn is manifested even in such an area as economics. Thus, The Economist magazine presented its annual rebus cover with a forecast for 2025, in the upper part of which Saturn with rings is depicted (Figure 1).

Figure 1 – A cover fragment of The Economist magazine with a forecast for 2025 with an image of Saturn

It can also be noted that in May 2025, the rings of Saturn are expected to "disappear". In fact, the rings will rotate in such a way (the observed ellipse will turn into a segment) that they will be practically invisible from the Earth. This happens regularly, twice per Saturnian year (approximately 29,5 Earth years), the previous time in 2009.

The rings of Saturn were studied by P. S. Laplace, S. V. Kovalevskaya, D. Maxwell, M. S. Bobrov and others [1–4].

In [5, 6], expressions were obtained for the intensity of the gravitational field of the ring for points of the plane in which the ring is located

$$g = -\frac{Gm}{\pi R^2} I(k)\,, \qquad (1)$$

where G – is the gravitational constant; m – is the mass of the ring; R – is the radius of the ring. The integral $I(k)$ in expression (1) has the form

$$I(k) = \int_0^{\pi} \frac{(k - \cos\alpha)\,d\alpha}{(1 + k^2 - 2k\cos\alpha)^{1,5}}\,. \tag{2}$$

For points located inside the ring, the parameter $k < 1$; for points located outside the ring the parameter $k > 1$.

As the point in question approaches the ring, the intensity of the gravitational field increases.

If the point is localized on the ring itself, then the parameter $k = 1$ and expression (2) are transformed to the form

$$I(k) = \frac{1}{2\sqrt{2}} \int_0^{\pi} \frac{d\alpha}{(1 - \cos\alpha)^{0,5}}\,. \tag{3}$$

The integral $I(k)$ in expression (3) is divergent. This leads to the fact that the magnitude of the gravitational field strength at points belonging to the ring tends to infinity [7]. This situation arises due to the idealization of the physical model, in which the mass is distributed along a circle, while the volumetric density of the mass tends to infinity.

This paper discusses methods for estimating the magnitude of the gravitational field strength at points localized on the ring itself, taking into account the finiteness of the volumetric mass density (Figure 2, *a*, point *O* in the cross section of the ring).

Estimation of the gravitational field strength of a ring-shaped mass distribution (for points belonging to the ring). First, consider a homogeneous solid infinite cylinder. The mass of this cylinder is the source of the gravitational field. The intensity lines of this field will be directed towards the axis of the cylinder. Figure 2, *b* shows a cross-section of the cylinder with stress lines.

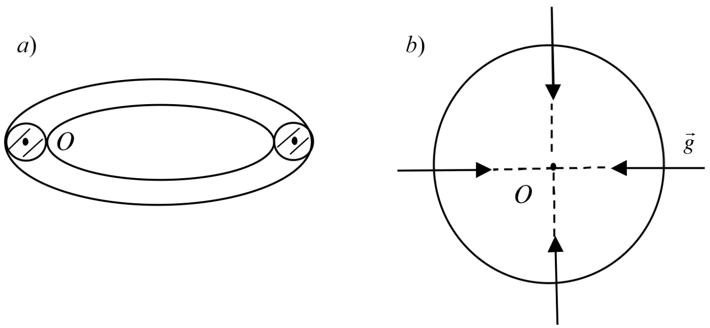

Figure 2 – Schemes: *a* – ring-shaped mass distribution with some bulk density; *b* – gravitational field intensity lines for a homogeneous cylinder

The dependence of the gravitational field strength on the distance to the cylinder axis is presented in Figure 3 (projection of the vector $\vec{g}$ for points along the *OX* axis).

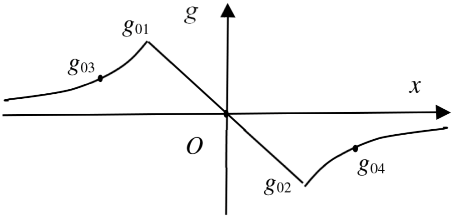

Figure 3 – Dependence of the gravitational field strength on the distance to the cylinder axis

The average value of the field strength is zero due to the fact that the strength has different signs to the left and right of point *O*. We can also say that at the center of the cylinder (on its axis) the field strength is zero due to symmetry.

If the cross section of the ring has a different shape, for example, as in Figure 4 then we can also assume that the field strength at the center is zero in the absence of other fields. And this same value can be considered the average value of the field strength over the cross section of the ring.

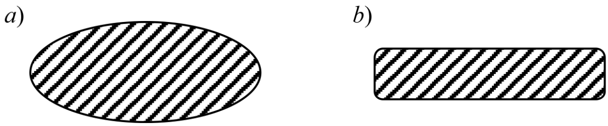

Figure 4 – Cross-section of the ring in the form of *a* – an ellipse or oval; *b* – a rectangle

In the presence of an external gravitational field, the graph in Figure 3 (for a cylinder) will change and take the form in Figure 5.

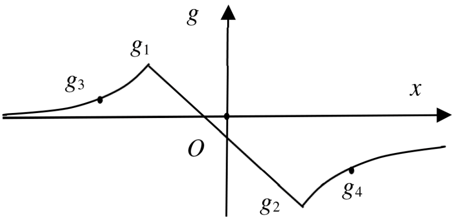

Figure 5 – Dependence of the gravitational field strength on the distance to the cylinder axis in the presence of an external gravitational field

In this case we can assume that

$$g_1 = g_{01} + g_{ex} \qquad \text{and} \qquad g_2 = g_{02} + g_{ex}, \tag{4}$$

where g_{ex} – is the intensity of the external gravitational field (we consider it constant).

Then from relations (4) we obtain

$$g_{ex} = \frac{g_1 + g_2 + g_{01} + g_{02}}{2} = \frac{g_1 + g_2}{2}. \tag{5}$$

Expression (5) takes into account that $g_{01} + g_{02} = 0$.

For other points (symmetric), we similarly obtain

$$g_{ex} = \frac{g_3 + g_4}{2}. \tag{6}$$

Now let's imagine the ring in question (see Figure 2, *a*) as a combination of two parts. The first part is a cylinder with a center at point *O*; its length is much greater than the cross section of the cylinder, but much less than the length $2\pi R$ of the ring. For point *O*, this part corresponds to the field of the cylinder. The second part is the rest of the ring. For point *O*, this part corresponds to the external field.

Then from (5) and (6) we can assume that the intensity of the gravitational field will be equal

$$g = \frac{g_1 + g_2}{2} \quad \text{or} \quad g = \frac{g_3 + g_4}{2}. \tag{7}$$

For points outside the ring, we take $k = 1 + \delta$, for points inside the ring we take $k = 1 - \delta$.

Then, taking into account expressions (1), (2), we can write

$$g_1 = -\frac{Gm}{\pi R^2} \int_0^{\pi} \frac{(1 - \delta - \cos\alpha)\, d\alpha}{\left(1 + (1-\delta)^2 - 2(1-\delta)\cos\alpha\right)^{1,5}}, \tag{8}$$

$$g_2 = -\frac{Gm}{\pi R^2} \int_0^{\pi} \frac{(1 + \delta - \cos\alpha)\, d\alpha}{\left(1 + (1+\delta)^2 - 2(1+\delta)\cos\alpha\right)^{1,5}}. \tag{9}$$

Accordingly, the gravitational field intensity created by the entire ring taking into account expressions (7)–(9):

$$g = -\frac{Gm}{2\pi R^2} \left[\int_0^{\pi} \frac{(1 - \delta - \cos\alpha)\, d\alpha}{\left(1 + (1-\delta)^2 - 2(1-\delta)\cos\alpha\right)^{1,5}} + \int_0^{\pi} \frac{(1 + \delta - \cos\alpha)\, d\alpha}{\left(1 + (1+\delta)^2 - 2(1+\delta)\cos\alpha\right)^{1,5}} \right]. \tag{10}$$

Let us introduce the notation

$$I(\delta) = \int_0^{\pi} \frac{(1 - \delta - \cos\alpha)\, d\alpha}{\left(1 + (1-\delta)^2 - 2(1-\delta)\cos\alpha\right)^{1,5}} + \int_0^{\pi} \frac{(1 + \delta - \cos\alpha)\, d\alpha}{\left(1 + (1+\delta)^2 - 2(1+\delta)\cos\alpha\right)^{1,5}},$$

then expression (10) will take the form

$$g = -\frac{Gm}{2\pi R^2} I(\delta) .$$

The values of the $I(\delta)$ for some values of δ are presented in Table 1.

Table 1 – **Values of the parameter $I(\delta)$ for some values of δ**

δ	$I(\delta)$	δ	$I(\delta)$
10^{-5}	5,9	10^{-3}	6,988
$3\cdot10^{-5}$	5,25	$3\cdot10^{-3}$	5,888
10^{-4}	9,25	10^{-2}	4,6855
$3\cdot10^{-4}$	8,19	$3\cdot10^{-2}$	3,5903

For estimation, we take the parameter $I(\delta) = 5$. Then the average value of the gravitational field strength at points belonging to the ring:

$$g = -\frac{5Gm}{2\pi R^2} . \tag{11}$$

Discussion and Conclusion. The average value of the gravitational field strength at points belonging to the ring (11) is proportional to the mass of the ring. It is also inversely proportional to the square of the radius of the ring. The structure of this expression is similar to the field strength of a ball or point mass [8–10].

The found average value of the gravitational field strength sets the magnitude of the force acting on the ring elementary mass *dm*. This force is directed towards the center of the ring and is equal to

$$dF \approx -\frac{5Gm}{2\pi R^2} dm .$$

The elemental mass *dm* can be expressed in terms of the ring length element *dl*:

$$dm = \frac{m}{2\pi R} dl .$$

From here we get

$$dF \approx -\frac{5Gm^2}{4\pi^2 R^3} dl .$$

In this expression the force is proportional to the square of the ring mass (without taking into account other gravitational fields). If there is a planet in the center of the ring, for example Saturn, then the force acting on the elementary section of the ring *dl* will be proportional to the product *Mm* (*M* is the mass of the planet). Since the mass M is much greater than the mass m, then the force influence of the planet will be dominant.

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ОПРЕДЕЛЕНИЕ НАПРЯЖЕННОСТИ ГРАВИТАЦИОННОГО ПОЛЯ КОЛЬЦЕОБРАЗНОГО РАСПРЕДЕЛЕНИЯ МАССЫ

Рассматривается случай кольцеобразного распределения массы. Исследуется способ оценки величины напряженности гравитационного поля в точках, локализованных на самом кольце, с учетом конечности объемной плотности массы. Получено выражение, позволяющее провести оценку величины напряженности гравитационного поля в точках, локализованных на самом кольце.

Ключевые слова: напряженность гравитационного поля, радиус кольца, масса кольца.

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